J. Fluid Mech. (1971), *vol.* 50, *part* 4, *pp.* 657–668

Printed in Great Britain

An exact periodic solution of the energy equation

By HIROSHI ISHIGAKI

National Aerospace Laboratory, Kakuda Branch, Miyagi, Japan

(Received 22 June 1971)

An exact periodic solution of the unsteady energy equation for an incompressible fluid with constant properties is derived to illustrate the effect of an oscillation through the viscous dissipation on a temperature field. The flow field used here is a generalization of the well-known Couette flow solution of steady flow, in which one wall is at rest and the other wall oscillates in its own plane about a constant mean velocity. The solution is subject to two boundary conditions that correspond to the heat-transfer and thermometer problems. In order to have some suggestions about the approximate solutions, the solution is compared with its own approximate form. The temperature field consists of a time-mean, first and second harmonic fluctuation. The time-mean temperature profiles show the large influence of oscillation. The time-mean heat flux into or the time-mean temperature of the oscillating wall increases with frequency, and is ultimately proportional to the square root of the frequency. In § 4 the present exact solution of the Couette flow is compared with the formerly obtained approximate solution of the flat plate boundary-layer flow in terms of the wall characteristic values at high frequencies.

1. Introduction

After the initiation by Lighthill (1954) there have been many works on the subject of laminar boundary layers which have a regular fluctuating flow superimposed on the mean flow. Owing to the mathematical difficulties most of them include restrictions on an oscillation amplitude or a frequency in the course of their theoretical developments. One of the exact solutions of the Navier–Stokes equation in which no restriction is placed on the amplitude and frequency was obtained by Stuart (1955) for the flow past a flat plate at zero incidence with uniform suction (asymptotic suction flow). With viscous dissipation of kinetic energy taken into account, the corresponding exact solution of the energy equation was also obtained by Stuart under a condition of zero heat transfer between the fluid and the wall. However, the heat-transfer problem was not studied.

The result of Stuart shows that the time-mean temperature of the wall rises with frequency and is ultimately proportional to the square root of the frequency. This effect was confirmed approximately by Ishigaki (1971 *a*) for the boundary-layer flow on a flat plate at zero incidence without suction (Blasius flow). In the succeeding paper of Ishigaki (1971 *b*) studying the corresponding heat-transfer problem it was approximately shown that the time-mean heat flux into the wall increases with frequency and is likewise ultimately proportional to the square root of the frequency. These studies imply that the main effect of the fluctuating velocity field on the time-mean temperature field may be the generation of heat through viscous dissipation, and we may encounter the practical problems of the thermal failure of a liquid rocket engine with a screaming combustion or of the resonance tube heating in which the effect of heat generation seems to be remarkable.

It is of some value to have an exact periodic solution of the unsteady energy equation to illustrate the large influence of the oscillation through the viscous dissipation on the temperature field. The purpose of this paper is to describe such a solution that also applies to the heat-transfer problem. A base for the periodic solution given here is the well-known steady Couette flow solution of the Navier–Stokes equation and the corresponding exact solution of the steady energy equation was obtained by Schlichting (1951). The solution is subject to two boundary conditions, that the moving wall is kept constant temperature or is insulated to heat.

2. Velocity field

We consider the two-dimensional flow of an incompressible fluid with constant properties between two parallel flat walls, one of which is at rest, the other moving in its own plane with an unsteady velocity. We restrict our considerations to the case in which the flow is independent of the distance along the wall and the velocity component normal to the wall is zero. Then the Navier–Stokes equation is written, in a co-ordinate system fixed with the wall moving with the velocity $-U(t)$, as

$$\left.\begin{aligned} \frac{\partial u}{\partial t} = \frac{dU}{dt} + \nu\frac{\partial^2 u}{\partial y^2}, \\ u = 0, \quad \text{at} \quad y = 0, \quad u = U(t) \quad \text{at} \quad y = h, \end{aligned}\right\} \tag{1}$$

in which y denotes the normal distance from the moving wall, u the velocity along the wall, t the time, ν the kinematic viscosity and h the distance between the two walls. We consider the case in which the unsteady velocity is given by

$$U(t) = U_h\{1 + \tfrac{1}{2}\epsilon(e^{i\omega t} + e^{-i\omega t})\}, \tag{2}$$

where ω is the frequency and U_h, ϵ are constants. We then look for a solution of the form

$$u = U_h\{f_0(\eta) + \tfrac{1}{2}\epsilon(f_1(\eta)\, e^{i\omega t} + \tilde{f}_1(\eta)\, e^{-i\omega t})\}, \tag{3}$$

in which $\eta = y/h$ and the tilde denotes a complex conjugate. Substituting (2), (3) into (1) and equating steady and periodic terms separately to zero, we have

$$\left.\begin{aligned} f_0'' = 0, \quad f_1'' - i\sigma f_1 = -i\sigma, \\ f_0 = f_1 = 0 \quad \text{at} \quad \eta = 0, \quad f_0 = f_1 = 1 \quad \text{at} \quad \eta = 1, \end{aligned}\right\} \tag{4}$$

where $\sigma = \omega h^2/\nu$ is the frequency parameter and primes denote differentiation with respect to η. The solutions are

$$f_0 = \eta, \quad f_1 = 1 - \cosh{(i\sigma)^{\frac{1}{2}}\eta} + \coth{(i\sigma)^{\frac{1}{2}}}.\sinh{(i\sigma)^{\frac{1}{2}}\eta}. \tag{5}$$

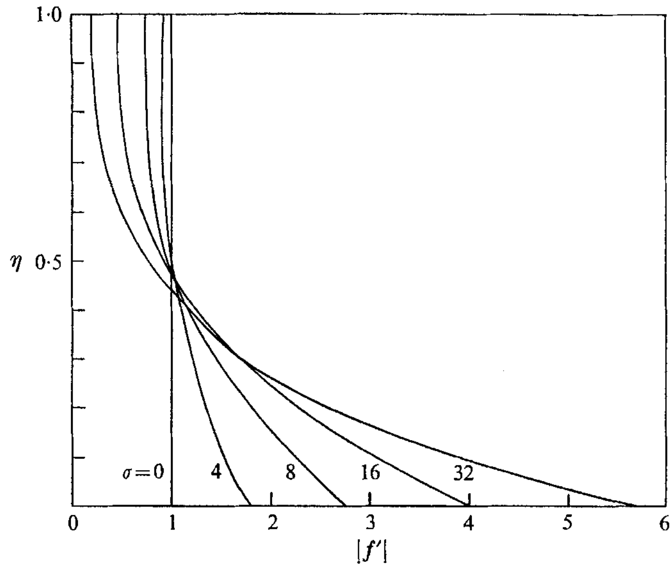

FIGURE 1. Profiles of fluctuating friction amplitude.

The distribution of the magnitude of fluctuating velocity gradient $|f_1'|$, which is directly related to the generation of heat, is shown in figure 1 for several values of σ. The shear stress at $y = 0$, $\tau_w = \mu(\partial u/\partial y)_{y=0}$, is found to be

$$\tau_w = (\mu U_h/h)\{1+\tfrac{1}{2}\epsilon(A\,e^{i\omega t}+\tilde{A}\,e^{-i\omega t})\}, \tag{6}$$

where $$A = (i\sigma)^{\frac{1}{2}}.\coth(i\sigma)^{\frac{1}{2}}.$$

This is readily compared with its own approximate form by expanding

$$A = \begin{cases} 1+\frac{1}{3}i\sigma-\frac{1}{45}(i\sigma)^2+\dots & \text{(small } \sigma) \quad (7\,a) \\ (i\sigma)^{\frac{1}{2}}+\dots & \text{(large } \sigma). \quad (7\,b) \end{cases}$$

The second term in (7*b*) is of order $(i\sigma)^{\frac{1}{2}}.\exp(-2(i\sigma)^{\frac{1}{2}})$ and decreases rapidly for large σ. The comparison between the exact expression and the combination

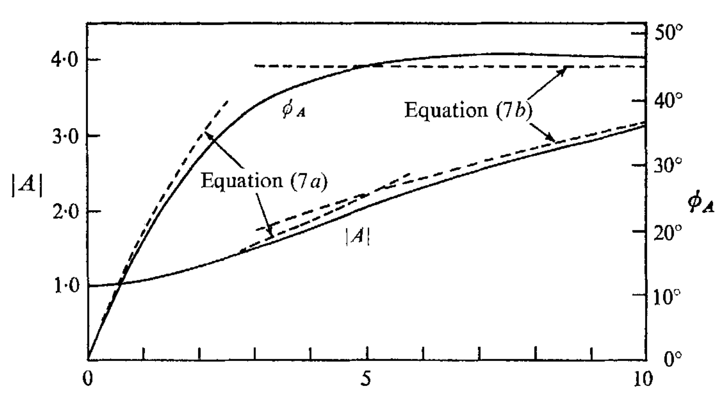

FIGURE 2. Skin friction amplitude and phase angle plotted against the frequency parameter.

of low- and high-frequency approximations is shown in figure 2 for the amplitude and phase angle of the fluctuating shear stress at $y = 0$. It can be found that the phase advance $\phi_A = \tan^{-1}(A_i/A_r)$, which is asymptotically given to be 45°, slightly overshoots at the intermediate values of σ.

3. Temperature field

Under the flow condition of § 2 the equation for the temperature distribution is given by

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x} = \kappa\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\frac{\nu}{c}\left(\frac{\partial u}{\partial y}\right)^2, \tag{8}$$

in which x denotes the distance along the wall, T the temperature, κ the thermal diffusivity and c the specific heat. Simple solutions are obtained when it is postulated that the temperature of the resting upper wall is constant, the temperature of the moving lower wall is constant (isothermal lower wall, say) or the lower wall is insulated to heat (adiabatic lower wall, say). Then the equation (8) is subject to the following boundary conditions:

$$T = T_w \quad \text{or} \quad \partial T/\partial y = 0 \quad \text{at} \quad y = 0, \quad T = T_h \quad \text{at} \quad y = h. \tag{9}$$

With these boundary conditions (8) has a solution which is independent of x. Then the convection term vanishes and the resulting temperature distribution is due to the generation of heat through friction and the conduction in the transverse direction. Thus we have

$$\frac{\partial T}{\partial t} = \kappa\frac{\partial^2 T}{\partial y^2}+\frac{\nu}{c}\left(\frac{\partial u}{\partial y}\right)^2. \tag{10}$$

Substituting (3) into (10), it can be seen that the exact solution for the temperature distribution is of the form

$$T(\eta) = T_0(\eta)+\tfrac{1}{2}\epsilon(T_1(\eta)\,e^{i\omega t}+\tilde{T}_1(\eta)\,e^{-i\omega t})+\tfrac{1}{2}\epsilon^2(T_2(\eta)\,e^{2i\omega t}+\tilde{T}_2(\eta)\,e^{-2i\omega t}). \tag{11}$$

Equating the harmonic coefficients to zero, we have

$$T_0'' = -(Pr\,U_h^2/c)\,(f_0'^2+\tfrac{1}{2}\epsilon^2 f_1'\tilde{f}_1'), \tag{12}$$

$$T_1''-iPr\,\sigma T_1 = -(Pr\,U_h^2/c)f_0'f_1', \tag{13}$$

$$T_2''-2iPr\,\sigma T_2 = -(Pr\,U_h^2/c)f_1'^2, \tag{14}$$

where $Pr = \nu/\kappa$ is the Prandtl number. There are two equations conjugate to (13) and (14). The boundary conditions are

$$\left.\begin{aligned} &T_0 = T_w, \quad T_1 = T_2 = 0, \quad \text{or} \quad T_0' = T_1' = T_2' = 0 \quad \text{at} \quad \eta = 0,\\ &T_0 = T_h, \quad T_1 = T_2 = 0 \qquad\qquad\qquad\qquad\qquad \text{at} \quad \eta = 1.\end{aligned}\right\} \tag{15}$$

Substituting (5) into (12), (13) and (14), we obtain the solutions which satisfy (15).

The solution of (12) for an isothermal lower wall gives the time-mean temperature distribution to be

$$(T_0-T_h)/(T_w-T_h) = 1-\eta+\tfrac{1}{2}Pr\,.\,Ec\{\eta-\eta^2+\epsilon^2(\tfrac{1}{2}(1-\eta)-\tfrac{1}{4}\alpha)\} \tag{16}$$

where $Ec = U_h^2/c(T_w - T_h)$ is the Eckert number and the symbol α is given by

$$\alpha = \cosh(2\sigma)^{\frac{1}{2}}\eta - \cos(2\sigma)^{\frac{1}{2}}\eta - (2/\beta)(\sinh(2\sigma)^{\frac{1}{2}}.\sinh(2\sigma)^{\frac{1}{2}}\eta + \sin(2\sigma)^{\frac{1}{2}}.\sin(2\sigma)^{\frac{1}{2}}\eta) + (1/\beta^2)(\sinh^2(2\sigma)^{\frac{1}{2}} + \sin^2(2\sigma)^{\frac{1}{2}}) \times (\cosh(2\sigma)^{\frac{1}{2}}\eta - \cos(2\sigma)^{\frac{1}{2}}\eta), \tag{17a}$$

in which
$$\beta = \cosh(2\sigma)^{\frac{1}{2}} - \cos(2\sigma)^{\frac{1}{2}}. \tag{17b}$$

The temperature distribution (16) consists of the following three terms: (i) linear distribution in the fluid at rest, (ii) parabolic distribution due to the frictional heating in the absence of oscillation, (iii) distribution due to the frictional heating caused by the wall oscillation. When the two walls have an equal temperature ($T_w = T_h$), heat conduction between the walls does not take place and (16) leads to

$$T_0 - T_h = (Pr\,U_h^2/2c)\{\eta - \eta^2 + \epsilon^2(\tfrac{1}{2}(1-\eta) - \tfrac{1}{4}\alpha)\}. \tag{18}$$

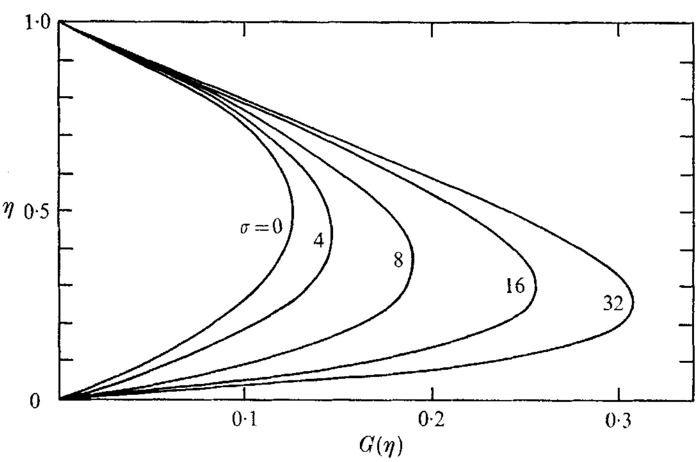

FIGURE 3. Plots of the frequency dependent function $G(\eta)$.

The frequency dependent part, $G(\eta) = \frac{1}{2}(1-\eta) - \frac{1}{4}\alpha$, is shown in figure 3 for several values of σ. When the oscillation is quasi-steady ($\sigma = 0$), the highest temperature created by the frictional heating occurs at the centre. It occurs nearer to the lower wall as σ becomes larger, being readily anticipated from the friction distribution in figure 1. For the case $Pr.Ec = 1{\cdot}0$ and $\epsilon = 2{\cdot}0$ the time-mean temperature distribution (16) is shown in figure 4 by solid lines, the dotted line in the figure being that without oscillation ($\epsilon = 0$).

The solution of (12) for an adiabatic lower wall gives the time-mean temperature distribution to be

$$T_0 - T_h = (Pr\,U_h^2/2c)[1 - \eta^2 + \epsilon^2\{(1/\beta)(1-\eta)(\tfrac{1}{2}\sigma)^{\frac{1}{2}} \times (\sinh(2\sigma)^{\frac{1}{2}} + \sin(2\sigma)^{\frac{1}{2}}) - \tfrac{1}{4}\alpha\}], \tag{19}$$

where α and β are already given by (17a, b). For the case $\epsilon = 1{\cdot}0$ the time-mean temperature distribution is shown in figure 5, the dotted line being that without oscillation. Figures 4 and 5 show that when the oscillation is present the generation of heat due to the friction exerts a large influence on the time-mean temperature field.

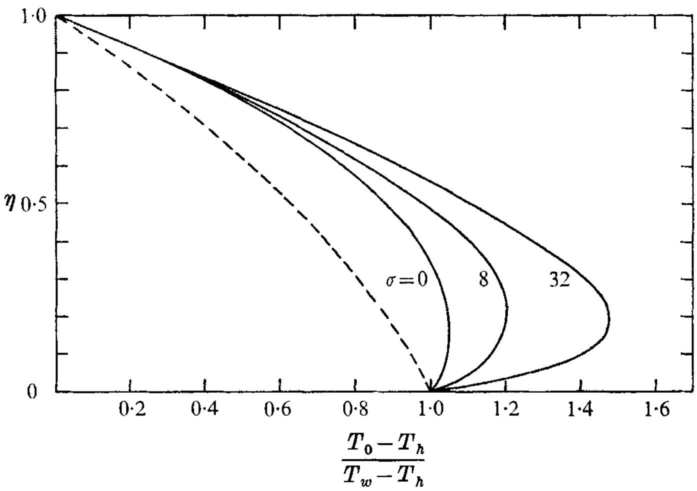

FIGURE 4. Temperature profiles for the case of the isothermal lower wall. $Pr\ Ec = 1{\cdot}0$; ——, $\epsilon = 2{\cdot}0$; – – –, $\epsilon = 0$.

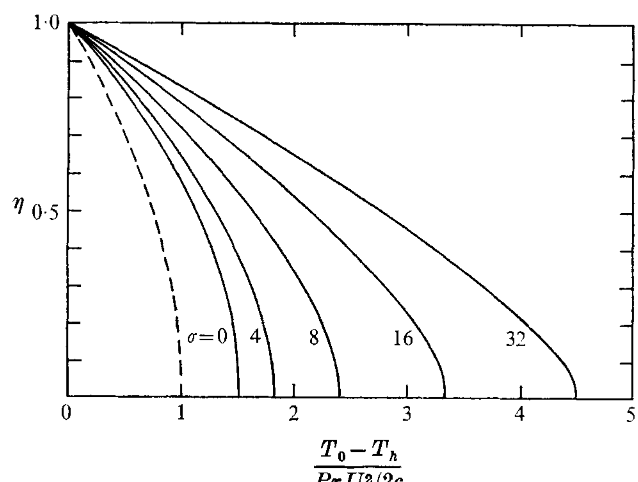

FIGURE 5. Temperature profiles for the case of an adiabatic lower wall. ——, $\epsilon = 1{\cdot}0$; – – –, $\epsilon = 0$.

In these cases the characteristic values at $y = 0$ are represented by

$$T_0'(0)/(T_w-T_h) = -1+\tfrac{1}{2}\,Pr\,Ec(1+\epsilon^2B(\sigma)) \quad \text{(isothermal lower wall)}, \tag{20}$$

$$T_0(0) = T_h+(Pr\,U_h^2/2c)\,(1+\epsilon^2B(\sigma)) \quad \text{(adiabatic lower wall)}, \tag{21}$$

where

$$B(\sigma) = (1/\beta)\,(\tfrac{1}{2}\sigma)^{\frac{1}{2}}\,(\sinh{(2\sigma)^{\frac{1}{2}}}+\sin{(2\sigma)})-\tfrac{1}{2}. \tag{22}$$

For small and large values of σ (22) takes the following forms:

$$B(\sigma) = \begin{cases} \tfrac{1}{2}+\tfrac{1}{45}\sigma^2+\dots & \text{(small } \sigma), \quad (23a) \\ (\tfrac{1}{2}\sigma)^{\frac{1}{2}}-\tfrac{1}{2}+\dots & \text{(large } \sigma). \quad (23b) \end{cases}$$

The frequency dependent function $B(\sigma)$ given by (22) is shown in figure 6 by a solid line and the approximate forms are shown by dotted lines, the first term only in (23*b*) being also shown by broken line. This comparison shows that the inclusion of the second term in the high-frequency approximation yields a more accurate formula, and this suggests that the corresponding high-frequency approximate solutions of the flat plate boundary-layer flow (Ishigaki 1971*a*, *b*) must be taken up to the second term.

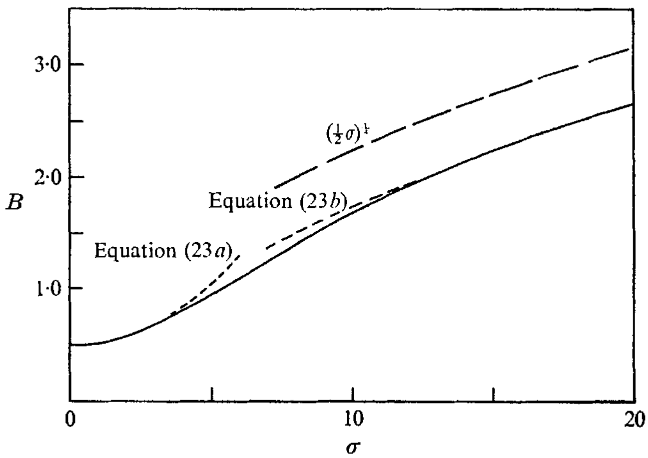

FIGURE 6. Plot of B against frequency parameter.

The solution of (13) is of the form

$$T_1 = (2Pr\, U_h^2/c)\,[c_1 \cosh\,(iPr\,\sigma)^{\frac{1}{2}}\eta + c_2 \sinh\,(iPr\,\sigma)^{\frac{1}{2}}\eta + (i\sigma)^{-\frac{1}{2}} \times \{\sinh\,(i\sigma)^{\frac{1}{2}}\eta - \coth\,(i\sigma)^{\frac{1}{2}}.\cosh\,(i\sigma)^{\frac{1}{2}}\eta\}] \quad (24)$$

and the coefficients are determined from the boundary conditions to be

$$\left.\begin{aligned} c_1 &= \frac{1}{(1-Pr)\,(i\sigma)^{\frac{1}{2}}}\coth\,(i\sigma)^{\frac{1}{2}}, \\ c_2 &= \frac{1}{(1-Pr)\,(i\sigma)^{\frac{1}{2}}}\{\operatorname{cosech}\,(i\sigma)^{\frac{1}{2}}.\operatorname{cosech}\,(iPr\,\sigma)^{\frac{1}{2}} - \coth\,(i\sigma)^{\frac{1}{2}}.\coth\,(iPr\,\sigma)^{\frac{1}{2}}\}\end{aligned}\right\} \quad (25)$$

for an isothermal lower wall, and

$$\left.\begin{aligned} c_1 &= \frac{1}{(1-Pr)\,(iPr\,\sigma)^{\frac{1}{2}}}\,(\tanh\,(iPr\,\sigma)^{\frac{1}{2}} + Pr^{\frac{1}{2}}\operatorname{cosech}\,(i\sigma)^{\frac{1}{2}}.\operatorname{sech}\,(iPr\,\sigma)^{\frac{1}{2}}), \\ c_2 &= -\frac{1}{(1-Pr)\,(iPr\,\sigma)^{\frac{1}{2}}}\end{aligned}\right\} \quad (26)$$

for an adiabatic lower wall. Then the characteristic wall value for an isothermal lower wall is given by

$$T_1'(0)/(T_w - T_h) = \tfrac{1}{2}Pr\,Ec\,C(\sigma), \quad (27a)$$

in which

$$C(\sigma) = (4/(1-Pr))\{1 + Pr^{\frac{1}{2}}(\operatorname{cosech}\,(i\sigma)^{\frac{1}{2}}.\operatorname{cosech}\,(iPr\,\sigma)^{\frac{1}{2}} - \coth\,(i\sigma)^{\frac{1}{2}}.\coth\,(iPr\,\sigma)^{\frac{1}{2}})\}. \quad (27b)$$

For small and large values of σ (27*b*) takes the following forms:

$$C = \begin{cases} 2+\frac{1}{6}(1-Pr)\,i\sigma-\frac{1}{60}(1-Pr^2)\,(i\sigma)^2+\dots & \text{(small } \sigma\text{)}, \quad (28a) \\ 4/(1+Pr^{\frac{1}{2}})+\dots & \text{(large } \sigma\text{)}. \quad (28b) \end{cases}$$

The wall value for an adiabatic lower wall is given by

$$T_1(0) = (Pr\,U_h^2/2c)\,.\,D(\sigma), \tag{29a}$$

in which

$$D(\sigma) = \frac{4}{(1-Pr)\,(iPr\,\sigma)^{\frac{1}{2}}}\{\tanh\,(iPr\,\sigma)^{\frac{1}{2}}+Pr^{\frac{1}{2}}(\operatorname{cosech}\,(i\sigma)^{\frac{1}{2}} \times \operatorname{sech}\,(iPr\,\sigma)^{\frac{1}{2}}-\coth\,(i\sigma)^{\frac{1}{2}})\}. \tag{29b}$$

For small and large values of σ (29*b*) takes the following forms:

$$D = \begin{cases} 2+\frac{1}{6}(1-5Pr)\,i\sigma-\frac{1}{180}(3+10Pr-61Pr^2)\,(i\sigma)^2+\dots & \text{(small } \sigma\text{)}, \quad (30a) \\ 4/(1+Pr^{\frac{1}{2}})\,(iPr\,\sigma)^{\frac{1}{2}}+\dots & \text{(large } \sigma\text{)}. \quad (30b) \end{cases}$$

For $Pr = 0{\cdot}72$ the amplitude and phase angle of first harmonic fluctuations are shown in figures 7 and 8 by solid lines, dotted lines denoting the approximate values calculated from (28) and (30). Figure 7 shows that the first harmonic fluctuation of heat transfer has a phase lead at low frequencies and has a phase lag at moderate frequencies, being in phase with the wall oscillation at high frequencies. Figure 8 shows that the first harmonic fluctuation of the adiabatic wall temperature always lags behind the wall oscillation and this phase lag approaches 45° at high frequencies.

The solution of (14) is of the form

$$\begin{aligned} T_2 = (U_h^2/8c)\,[&c_3\cosh\,(2iPr\,\sigma)^{\frac{1}{2}}\,.\,\eta+c_4\sinh\,(2iPr\,\sigma)^{\frac{1}{2}}\,.\,\eta+\coth^2(i\sigma)^{\frac{1}{2}}-1 \\ &-(Pr/(2-Pr))\,\{(\coth^2(i\sigma)^{\frac{1}{2}}+1)\cosh\,(2i\sigma)^{\frac{1}{2}}\,.\,\eta-2\coth\,(i\sigma)^{\frac{1}{2}}\,.\sinh\,(2i\sigma)^{\frac{1}{2}}\,.\,\eta\}], \end{aligned} \tag{31}$$

and the coefficients are determined from the boundary conditions to be

$$\begin{aligned} c_3 &= 1-\coth^2(i\sigma)^{\frac{1}{2}}+(Pr/(2-Pr))\,(1+\coth^2(i\sigma)^{\frac{1}{2}}), \\ c_4 &= (\coth^2(i\sigma)^{\frac{1}{2}}-1)\,(\coth\,(2iPr\,\sigma)^{\frac{1}{2}}-\operatorname{cosech}\,(2iPr\,\sigma)^{\frac{1}{2}})+(4Pr/(Pr-2)) \\ &\quad\times\cosh^2(i\sigma)^{\frac{1}{2}}\,.\operatorname{cosech}\,(2iPr\,\sigma)^{\frac{1}{2}}+(Pr/(Pr-2))\,(\coth^2(i\sigma)^{\frac{1}{2}}+1) \\ &\quad\times(\coth\,(2iPr\,\sigma)^{\frac{1}{2}}-\cosh\,(2(i\sigma)^{\frac{1}{2}})\operatorname{cosech}\,(2iPr\,\sigma)^{\frac{1}{2}}), \end{aligned} \tag{32}$$

for an isothermal lower wall and

$$\begin{aligned} c_3 &= (1-\coth^2(i\sigma)^{\frac{1}{2}})\,.\operatorname{sech}\,(2iPr\,\sigma)^{\frac{1}{2}}+(2(2Pr)^{\frac{1}{2}}/(2-Pr))\coth\,(i\sigma)^{\frac{1}{2}} \\ &\quad\times\tanh\,(2iPr\,\sigma)^{\frac{1}{2}}+(Pr/(2-Pr))\,\{(\coth^2(i\sigma)^{\frac{1}{2}}+1)\cosh\,(2(i\sigma)^{\frac{1}{2}}) \\ &\quad-4\cosh^2(i\sigma)^{\frac{1}{2}}\}\operatorname{sech}\,(2iPr\,\sigma)^{\frac{1}{2}}, \\ c_4 &= (2(2Pr)^{\frac{1}{2}}/(Pr-2))\coth\,(i\sigma)^{\frac{1}{2}}, \end{aligned} \tag{33}$$

for adiabatic lower wall. Then the characteristic wall value for isothermal lower wall is given as

$$T_2'(0)/(T_w-T_h) = \tfrac{1}{2}Pr\,.\,Ec\,.\,E(\sigma), \tag{34a}$$

where

$$E(\sigma) = \frac{(2i\sigma)^{\frac{1}{2}}}{4}\,.\left(\frac{c_4}{Pr^{\frac{1}{2}}}+\frac{1}{2-Pr}\coth\,(i\sigma)^{\frac{1}{2}}\right), \tag{34b}$$

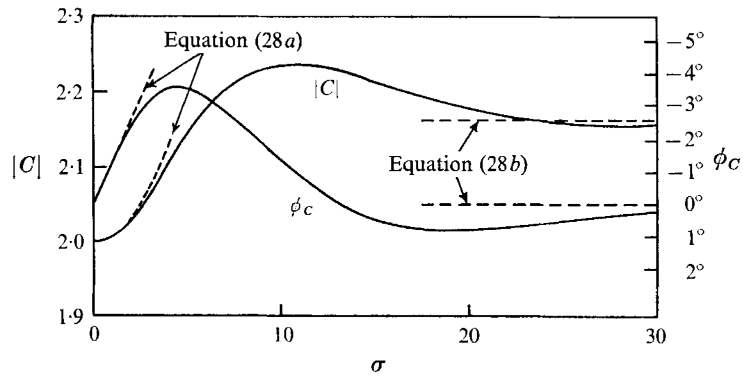

FIGURE 7. Amplitude and phase angle of the first harmonic fluctuation of heat transfer, $Pr = 0{\cdot}72$.

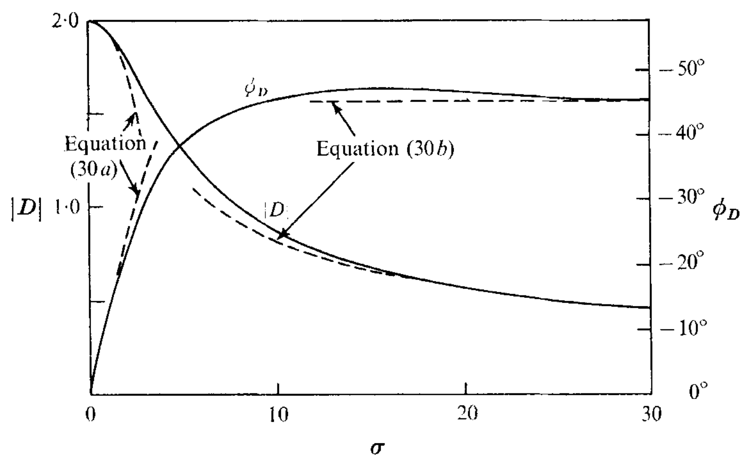

FIGURE 8. Amplitude and phase angle of the first harmonic fluctuation of adiabatic wall temperature, $Pr = 0{\cdot}72$.

c_4 being given by (32). For small and large values of σ, (34*b*) takes the following forms:

$$E = \tfrac{1}{2}+\tfrac{1}{12}(1-Pr)\,i\sigma+\tfrac{1}{360}(11Pr^2+55Pr-58)\,(i\sigma)^2+\dots \quad \text{(small } \sigma), \quad (35a)$$

$$= (i\sigma)^{\frac{1}{2}}/2^{\frac{1}{2}}(2^{\frac{1}{2}}+Pr^{\frac{1}{2}})+\dots \quad \text{(large } \sigma). \quad (35b)$$

The wall value for an adiabatic lower wall is given as

$$T_2(0) = (Pr\,U_h^2/2c)\,F(\sigma), \quad (36a)$$

where $$F(\sigma) = \frac{1}{4Pr}\left\{c_3+\coth^2(i\sigma)^{\frac{1}{2}}-1-\frac{Pr}{2-Pr}(\coth^2(i\sigma)^{\frac{1}{2}}+1)\right\}, \quad (36b)$$

c_3 being given by (33). For small and large values of σ, (36*b*) takes the following forms:

$$F = \tfrac{1}{2}+\tfrac{1}{12}(1-5Pr)\,i\sigma+\tfrac{1}{180}(61Pr^2-10Pr+1)\,(i\sigma)^2+\dots \quad \text{(small } \sigma), \quad (37a)$$

$$= 1/2Pr^{\frac{1}{2}}(2^{\frac{1}{2}}+Pr^{\frac{1}{2}})+\dots \quad \text{(large } \sigma). \quad (37b)$$

For $Pr = 0{\cdot}72$ the amplitude and phase angle of the second harmonics of the wall values are shown in figures 9 and 10 by solid lines, approximate values being shown by dotted lines. In figure 10 the considerable phase lag of the adiabatic wall temperature fluctuation at lower frequencies is noted.

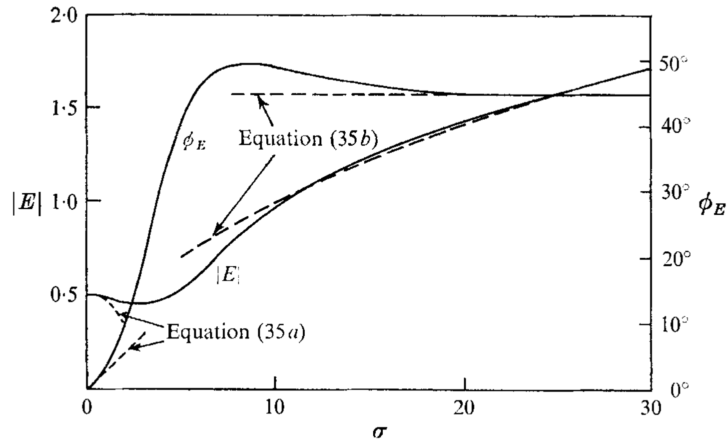

FIGURE 9. Amplitude and phase angle of the second harmonic fluctuation of heat transfer, $Pr = 0{\cdot}72$.

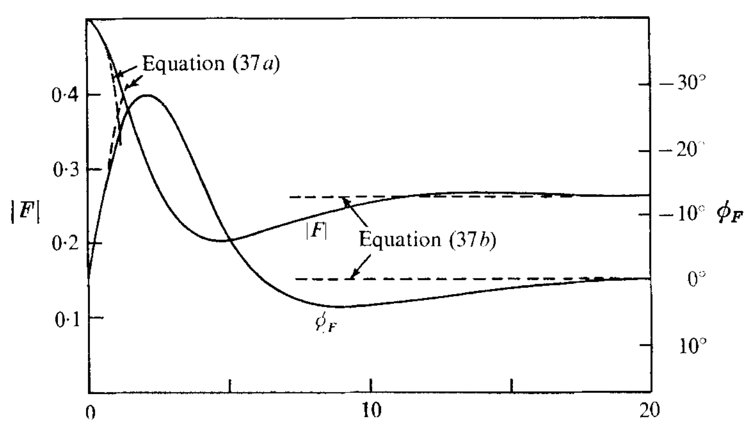

FIGURE 10. Amplitude and phase angle of the second harmonic fluctuation of adiabatic wall temperature, $Pr = 0{\cdot}72$.

4. Discussion

The results obtained in § 3 illustrate the influence of oscillation through the viscous dissipation on the temperature field. At high frequencies the time-mean and second harmonic components of the temperature field are greatly influenced. In particular, the time-mean effect is of practical importance for the considerations of cooling problems and of fluid temperature measurement in the presence of intense flow oscillation of high frequency. Thus a note on the time-mean results may be helpful. When the lower wall temperature is higher than the upper wall temperature ($T_w > T_h$), heat flows from the lower wall to the fluid only when the parameters $Pr.Ec$, ϵ and σ do not exceed certain values. A reversal of heat flow direction at the lower wall occurs when the temperature gradient there changes sign. Thus it is seen from (20) that the following inequality applies to the condition of heat flowing from the lower wall to the fluid:

$$2 > Pr.Ec(1+\epsilon^2 B(\sigma)). \tag{38a}$$

The above criterion for cooling of the lower wall is simplified if the adiabatic wall temperature T_a defined by (21) is introduced. We then have

$$T_w > T_a. \tag{38b}$$

When $T_h > T_w$, the heat flux into the lower wall increases greatly because the heat generated by friction is superimposed on the conducting heat, and we can see a practical example in the drastic increase of heat flux from hot gas to engine wall when high-frequency oscillatory combustion occurs in liquid rocket engines.

Next, it is intended to compare the present results with the other available results of Blasius flow and asymptotic suction flow. In the problem of an incompressible periodic flow, the thickness of the Stokes layer or the penetration depth, $\delta_0 \sim (\nu/\omega)^{\frac{1}{2}}$, plays the role of a boundary-layer thickness. When a steady stream is present the original unperturbed boundary layer (thickness δ_1) and the Stokes layer co-exist and interact, the square of the thickness ratio, $(\delta_1/\delta_0)^2$, being equivalent to the frequency parameter. When the frequency parameter is large the essential character of the flow field will be of the Stokes type regardless of the original boundary layer. Therefore we make comparison of the temperature fields in terms of the asymptotic expressions of wall values for high frequency. If we extract only the terms associated with oscillation the present exact solution of Couette flow for an isothermal lower wall gives the asymptotic form to be

$$T'(0) \sim \frac{Pr\,U_h^2}{2c}\left[\epsilon^2(\tfrac{1}{2}\sigma)^{\frac{1}{2}}+\frac{4\epsilon}{1+Pr^{\frac{1}{2}}}\cos\omega t+\frac{\epsilon^2\sigma^{\frac{1}{2}}}{2^{\frac{1}{2}}+Pr^{\frac{1}{2}}}.\cos(2\omega t+\tfrac{1}{4}\pi)\right], \tag{39}$$

and

$$T(0) \sim \frac{U_h^2}{2c}\left[\epsilon^2 Pr(\tfrac{1}{2}\sigma)^{\frac{1}{2}}+\frac{4\epsilon}{1+Pr^{\frac{1}{2}}}\left(\frac{Pr}{\sigma}\right)^{\frac{1}{2}}.\cos(\omega t-\tfrac{1}{4}\pi)+\frac{\epsilon^2 Pr^{\frac{1}{2}}}{2(2^{\frac{1}{2}}+Pr^{\frac{1}{2}})}\cos 2\omega t\right]. \tag{40}$$

for an adiabatic lower wall. The approximate solution of Blasius flow in which the outer flow velocity is given by $U_\infty(1+\epsilon\cos\omega t)$ gives the temperature gradient at the wall, $T_p'(0)$ for an isothermal flat plate to be (Ishigaki 1971*b*)

$$T_p'(0) \sim \frac{Pr\,U_h^2}{2c}\left[\epsilon^2\sigma_p^{\frac{1}{2}}+\frac{4\epsilon f''(0)}{1+Pr^{\frac{1}{2}}}\cos\omega t+\frac{\epsilon^2\sigma_p^{\frac{1}{2}}}{2^{\frac{1}{2}}+Pr^{\frac{1}{2}}}\cos(2\omega t+\tfrac{1}{4}\pi)\right], \tag{41}$$

in which U_∞ is constant, $\sigma_p = \omega x/U_\infty$ (x is the distance along the wall from the leading edge) and $f''(0) = 1{\cdot}2326$. The corresponding approximate solution for an adiabatic flat plate gives the wall temperature $T_p(0)$ to be (Ishigaki 1971*a*)

$$T_p(0) \sim \frac{U_\infty^2}{2c}\left[\epsilon^2 P_0(0)\,\sigma_p^{\frac{1}{2}}+\frac{4\epsilon f''(0)}{1+Pr^{\frac{1}{2}}}\left(\frac{Pr}{\sigma_p}\right)^{\frac{1}{2}}.\cos(\omega t-\tfrac{1}{4}\pi)+\frac{\epsilon^2 Pr^{\frac{1}{2}}}{2(2^{\frac{1}{2}}+Pr^{\frac{1}{2}})}\cos 2\omega t\right], \tag{42}$$

in which $P_0(0)$ is the function of Pr only. The exact solution for asymptotic suction flow in which the outer flow velocity is given by $U_s(1+\epsilon\cos\omega t)$ gives the adiabatic wall temperature $T_s(0)$ to be (Stuart 1955)

$$T_s(0) \sim \frac{U_s^2}{2c}\left[\epsilon^2(\tfrac{1}{2}\sigma_s)^{\frac{1}{2}} + \frac{4\epsilon}{1+Pr^{\frac{1}{2}}}\left(\frac{Pr}{\sigma_s}\right)^{\frac{1}{2}}\cos(\omega t-\tfrac{1}{4}\pi) + \frac{\epsilon^2 Pr^{\frac{1}{2}}}{2(2^{\frac{1}{2}}+Pr^{\frac{1}{2}})}\cos 2\omega t\right], \quad (43)$$

in which U_s is constant and $\sigma_s = \omega\nu/v_w^2$ (v_w is the constant suction velocity). As the unperturbed original boundary-layer thickness δ_1 is proportional to $(\nu x/U_\infty)^{\frac{1}{2}}$ in Blasius flow and to ν/v_w in asymptotic suction flow, the frequency parameters σ_p and σ_s are proportional to $(\delta_1/\delta_0)^2$, σ being proportional to $(h/\delta_0)^2$. (The detailed discussion on the frequency parameter is given by Stuart (1963).) Thus we can see the correspondences of the magnitude and phase angle between (39) and (41) or among (40), (42) and (43). Expressions (39), (40) and (43) are the exact results and do not contain harmonics higher than the second. Equations (41) and (42) do contain them, but they are expected to be of a much smaller order of magnitude for high-frequency oscillation.

REFERENCES

ISHIGAKI, H. 1971*a* *J. Fluid Mech.* **46**, 165.
ISHIGAKI, H. 1971*b* *J. Fluid Mech.* **47**, 537.
LIGHTHILL, M. J. 1954 *Proc. Roy. Soc.* A **224**, 1.
SCHLICHTING, H. 1951 *Z. angew. Math. Mech.* **31**, 78.
STUART, J. T. 1955 *Proc. Roy. Soc.* A **231**, 116.
STUART, J. T. 1963 In *Laminar Boundary Layers* (ed. L. Rosenhead), p. 347. Oxford University Press.